Two Functional Analytic Fixed Point Theorems

Daniel Raban

November 18, 2019

1 Fixed Point Theorems

Fixed point theorems are theorems that do exactly what they say on the box; if we have a map (or family of maps) T, they guarantee the existence of a point x such that $Tx = x$ (for all T simultaneously). The general use of fixed point theorems is to prove existence of some object we want; we then come up with maps T such that $Tx = x$ gives x the property we want.

These notes will cover two fixed point theorems and briefly mention an application of each.

1.1 The Markov-Kakutani fixed point theorem

Definition 1.1. Let X be a vector space, and let $K \subseteq X$ be convex. An operator $T : K \to K$ is called **affine** if

$$T(tx_1 + (1-t)x_2) = tTx_1 + (1-t)Tx_2, \qquad x_1, x_2 \in X, t \in [0,1].$$

In other words, T commutes with taking weighted averages of elements in K. This is like a weaker version of linearity that corresponds to compatibility with convex sets.

Theorem 1.1 (Markov-Kakutani)**.** *Let X be a LCS*[1]*, let $K \subseteq X$ be nonempty, compact, and convex, and let $\mathcal{F}$ be a family of mutually commuting, continuous, affine maps $K \to K$. Then there exists some $x_0 \in K$ such that $Tx_0 = x_0$ for all $T \in \mathcal{F}$.*

[1] This means a locally convex vector space. If you don't know what this is, it's a slight generalization of vector spaces with a norm topology.

Proof. For each $T \in \mathcal{F}$, define $T^{(n)} = \frac{1}{n}\sum_{k=0}^{n-1} T^k$. Then $T^{(n)}$ is a again an affine map taking $K \to K$. If $S, T \in \mathcal{F}$, then $S^{(n)}, T^{(m)}$ commute for all n, m. Let $\mathcal{K} = \{T^{(n)}(K) : T \in \mathcal{F}, n \geq 1\}$, which is a collection of compact, convex sets. If $T_1, \dots, T_p \in \mathcal{F}$ and $n_1, \dots, n_p \geq 1$, then

$$T_1^{(n_1)} \circ \cdots \circ T_p^{(n_p)}(K) \subseteq \bigcap_{j=1}^{p} T_j^{(n_j)}(K).$$

Then $\mathcal{K}$ has the finite intersection property. So there exists an $x_0 \in \bigcap_{K' \in \mathcal{K}} K'$.

We claim that x_0 is the desired fixed point. Take $t \in \mathcal{F}$, and let $n \geq 1$. Then $x_0 \in T^{(n)}(K)$, so $x_0 = T^{(n)}(x)$ for some x. In particular,

$$x_0 = \frac{1}{n}(x + T(x) + \cdots + T^{n-1}(x)).$$

Applying T, we get

$$T(x_0) = \frac{1}{n}(T(x) + \cdots + T^{n-1}(x) + T^n(x)).$$

Subtracting this, we get

$$T(x_0) - x_0 = \frac{1}{n}(T^n(x) - x) \in \frac{1}{n}(K - K),$$

where $K - K$ is compact. This is true for any n. If U is an open neighborhood of 0, then there exists some n such that $\frac{1}{n}(K - K) \subseteq U$. Then $T(x_0) - x_0 \in U$ for all open neighborhoods U of 0, so $T(x_0) = x_0$. □

Here is an application.

Definition 1.2. Let G be a compact topological group. A **(left) Haar measure** is a probability measure μ on G such that $g_*\mu = \mu$ for all $g \in G$ (i.e. $\mu(gA) = \mu(A)$ for all measurable A and $g \in G$).

Corollary 1.1. *Every compact, abelian group possesses a (left) Haar measure.*

Proof. By the Riesz representation theorem and the Banach-Alaoglu theorem, $\mathcal{P}(G)$, the probability measures on G, form a convex, weak*-compact subset of $C(G)^* \cong M(G)$. Given $g \in G$, we can define $T_g : C(G)^* \to C(G)^*$ given by $T_g\mu \mapsto g_*\mu$. Then $\mathcal{F} := \{T_g : g \in G\}$ is a family of mutually commuting, weak*-continuous, affine maps $\mathcal{P}(G) \to \mathcal{P}(G)$. By the Markov-Kakutani fixed point theorem, there exists a fixed point μ such that $g_*\mu = \mu$ for all $g \in G$. □

1.2 The Ryll-Nardzewski fixed point theorem

Definition 1.3. Let X be a LCS, and let $E \subseteq X$. A family $\mathcal{F}$ of maps $E \to E$ is called **noncontracting** if for any $x \neq y \in E$, there exists a continuous seminorm p on X such that

$$\inf_{T \in \mathcal{F}} p(Tx - Ty) > 0.$$

So a family is noncontracting iff it separates points with respect to a seminorm. Here is the Ryll-Nardzewski fixed point theorem.

Theorem 1.2 (Ryll-Nardzewski)**.** *Let X be a LCS, let $K \subseteq X$ be nonempty, weakly compact, and convex, and let $\mathcal{F}$ be a noncontracting semigroup of weakly continuous affine maps $K \to K$. Then there is a point x_0 such that $Tx_0 = x_0$ for all $T \in \mathcal{F}$.*

The following application of the Ryll-Nardzewski fixed point theorem is a strengthening of our previous application:

Corollary 1.2. *Every compact group possesses a (left) Haar measure.*

Proof. The proof is the same as before, except we need to verify that the maps $T_g : \mathcal{P}(G) \to \mathcal{P}(G)$ sending $\mu \mapsto g_* \mu$ are noncontracting. This follows because each T_g is an isometry. $\square$

To prove the theorem, we need a lemma.

Lemma 1.1 (Namioka-Asplund)**.** *Let X be a LCS, let $Q \subseteq X$ be nonempty, separable, weakly compact, and convex, and let p be a continuous seminorm on X. Then for any $\varepsilon > 0$, there exists a closed, convex subset $C \subseteq Q$ such that*

$$C \neq Q, \qquad \sup_{x,y \in Q \setminus C} p(x-y) < \varepsilon.$$

In other words, if we have a compact convex set, we can shave off a piece of it with small p-diameter and stay closed and convex.

Now let's prove the theorem.

Proof. We first show that any finite subfamily $\{T_1, \dots, T_n\} \subseteq \mathcal{F}$ has a fixed point; we will later upgrade this result using compactness. Define

$$T_0 = \frac{1}{n}(T_1 + \cdots + T_n).$$

By the Markov-Kakutani fixed point theorem, there exists some x_0 with $T_0 x_0 = x_0$. For contradiction, assume there exists some $1 \le k \le n$ such that $T_k x_0 \ne x_0$; then we can reorder the maps T_i to ensure that

$$T_k x_0 \ne x_0 \quad \forall 1 \le k \le m, \qquad T_k x_0 = x_0 \quad \forall m+1 \le k \le n.$$

Now denote

$$T_0' = \frac{1}{m}(T_1 + \cdots + T_m).$$

Then $T_0' x_0 = x_0$, as well:

$$\begin{aligned} T_0' x_0 &= \frac{n}{m} \cdot \frac{1}{n}(T_1 x_0 + \cdots + T_m x_0) \\ &= \frac{n}{m} \cdot \left(T_0 x_0 - \frac{1}{n}(T_{m+1} x_0 + \cdots + T_n x_0) \right) \\ &= \frac{n}{m} \cdot \left(x_0 - \frac{n-m}{n} x_0 \right) \\ &= x_0. \end{aligned}$$

Since $\mathcal{F}$ is noncontracting, there exists a continuous seminorm p on X and an $\varepsilon > 0$ such that

$$p(TT_k x - Tx_0) \ge 2\varepsilon \qquad \forall T \in \mathcal{F}, 1 \le k \le m.$$

Indeed, the noncontracting property gives us a seminorm that works for a single k, and we can take the sum of these. Now let's get into position to use the lemma.

Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be the subsemigroup generated by $T_1, \ldots, T_n$. That is,

$$\mathcal{F}_0 = \{T_{i_1} \cdots T_{i_m} : 1 \le i_j \le n\}.$$

This is countable. Put $Q = \overline{\text{co}}\{Tx_0 : T \in \mathcal{F}_0\}$; then Q is

- weakly compact: as a closed subset of the weakly compact set K,
- separable: by the countability of the generating set.

By the lemma, there exists a closed, convex subset $C \subseteq Q$ such that $C \ne Q$ and

$$\sup_{x,y \in Q \setminus C} p(x - y) < \varepsilon.$$

Since $C \neq Q$, there exists some $S \in \mathcal{F}_0$ such that $Sx_0 \in Q \setminus C$. So we get

$$Sx_0 = ST_0x_0 = \frac{1}{m}(ST_1x_0 + \cdots + ST_1x_0) \in Q \setminus C.$$

By the convexity of C, since this average is not in C, there must be some k such that $ST_kx_0 \notin C$. But then

$$p(S(T_kx_0) - Sx_0) \leq p\text{-diam}(Q \setminus C) \leq \varepsilon,$$

which contradicts our choice of seminorm p.

We have shown that every finite subfamily of maps in $\mathcal{F}$ has a fixed point. For each $T \in \mathcal{F}$, define its set of fixed points $FP(T) = \{x \in K : Tx = x\}$. Then $FP(T)$ is

- weakly compact: because T is weakly continuous and K is weakly compact.

So by the finite case, we have shown that $\{FP(T) : T \in \mathcal{F}\}$ is a family of weakly compact sets with the finite intersection property. So the intersection $\bigcap_{T \in \mathcal{F}} FP(T)$ is nonempty. Any element of this intersection must be a mutual fixed point of all the $T \in \mathcal{F}$. $\square$

References

[Con94] J.B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer New York, 1994.

[SZ19] S. Stratila and L. Zsido. *Lectures on von Neumann Algebras*. Cambridge IISc Series. Cambridge University Press, 2019.